# The Quantum Lobachevsky Space and the $q$-Bessel-Macdonald Functions

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## 1 Classical case

Let $\mathbf{L}^3 = SU_2\backslash SL_2(\mathbf{C})$be a homogeneous space of the second-order unimodular Hermitian positive definite matrices, which is a model of the classical Lobachevsky space. Let

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma = 1.$$

Then any $x \in \mathbf{L}^3$ can be represented as

$$x = g^\dagger g = \begin{pmatrix} \bar\alpha\alpha + \bar\gamma\gamma & \bar\alpha\beta + \bar\gamma\delta \\ \bar\beta\alpha + \bar\delta\gamma & \bar\beta\beta + \bar\delta\delta \end{pmatrix}. \tag{1.1}$$

The Iwasawa decomposition

$$g = kb, \quad g \in SL_2(\mathbf{C}), \quad k \in SU_2, \quad b \in AN, \tag{1.2}$$

$AN$ - Borel subgroup, allows us to define the horospherical coordinates on $\mathbf{L}^3$. If

$$b = \begin{pmatrix} h & hz \\ 0 & h^{-1} \end{pmatrix},$$

then from (1.1)

$$x = b^\dagger b = \begin{pmatrix} \bar h h & \bar h h z \\ 0 & \bar z \bar h h z + (\bar h h)^{-1} \end{pmatrix}. \tag{1.3}$$

The tripl $(H + \bar h h, z, \bar z)$ is uniquely determined by $x$. It is called the horospherical coordinates of $x$. It follows from (1.1) and (1.3) that

$$H = \bar\alpha\alpha + \bar\gamma\gamma, \quad Hz = \bar\alpha\beta + \bar\gamma\delta, \quad \bar z H = \bar\beta\alpha + \bar\delta\gamma.$$

Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

be the generators of the Lie algebra $gl_2$ and $d_A, d_B, d_C$ and $d_D = -d_A$ be the corresponding Lie operators of right shift on $\mathbf{L}^3$. In the horospherical coordinates they take the form

$$d_A = \frac{1}{2}H\partial_H - z\partial_z, \quad d_B = \partial_z, \quad d_C = Hz\partial_H - z^2\partial_z + H^{-2}\partial_{\bar z}. \tag{1.4}$$

The cecond Casimir

$$\Omega = d_A^2 + d_D^2 + d_B d_C + d_C d_B$$

in the horospherical coordinates takes the form

$$\Omega = \frac{1}{2}H^2\partial_H^2 + \frac{3}{2}H\partial_H + 2H^{-2}\partial_{\bar z z}^2. \tag{1.5}$$

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Consider the eigenvalue problem

$$[\frac{1}{2}\Omega + \frac{1}{4}]F_\nu(\bar{z}, H, z) = \frac{\nu^2}{4}F_\nu(\bar{z}, H, z), \quad \nu > 0. \tag{1.6}$$

After the Fourier transform with respect the variables $\bar{z}$ and $z$ we have the ordinary differential equation for the Fourier image of $F_\nu(\bar{z}, H, z)$

$$(\frac{1}{4}H^2\frac{d^2}{dH^2} + \frac{3}{4}H\frac{d}{dH} - H^{-2}\bar{s}s + \frac{1}{4})\Phi_\nu(\bar{s}, H, s) = \frac{\nu^2}{4}\Phi_\nu(\bar{s}, H, s). \tag{1.7}$$

The solutions to equation (1.7) decreasing for $H \to 0$ are the functions

$$\Phi_\nu(\bar{s}, H, s) = \frac{\pi}{\Gamma(\nu+1)}H^{-1}K_\nu(2\sqrt{\bar{s}s}H^{-1})(\bar{s}s)^{\frac{\nu}{2}}\phi(\bar{s}, s), \tag{1.8}$$

where $K_\nu$ is the Bessel-Macdonold function, and $\phi_\nu(\bar{s}, s)$ is determined uniqualy by $\Phi_\nu(\bar{s}, H, s)$. It is well-known fact that

$$\frac{\pi}{\Gamma(\nu+1)}H^{-1}K_\nu(2\sqrt{\bar{s}s}H^{-1})(\bar{s}s)^{\frac{\nu}{2}}$$

is the Fourier transform of the function

$$P_\nu(\bar{z}, H, z) = (\bar{z}Hz + H^{-1})^{-\nu-1}. \tag{1.9}$$

After the inverse Fourier transform we obtain the solution to equation (1.7) in form

$$F_\nu(\bar{z}, H, z) = P_\nu(\bar{z}, H, z) * f(\bar{z}, z), \tag{1.10}$$

where $f(\bar{z}, z)$ is the inverse Fourier image of $\phi(\bar{s}, s)$.

Function (1.9) is called the Poisson kernel, and convolution (1.10) is called the Poisson integral.

## 2 Quantum Lobachevsky Space

Let $\mathcal{A}_q(SL_2(\mathbf{C})), q \in (0;1)$, be the algebra of functions on $SL_2(\mathbf{C})$ [2], which is defined as the factor algebra of the associate $\mathbf{C}$-algebra with generators $\alpha, \beta, \gamma, \delta$ with an anti-involution $^* : \mathcal{A}_q \to \mathcal{A}_q, (ab)^* = b^*a^*$ and the following relations

$$\begin{gathered}\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \quad \beta\gamma = \gamma\beta,\\ \alpha\delta - q\beta\gamma = 1, \quad \delta\alpha - q^{-1}\beta\gamma = 1, \quad \beta\alpha^* = q^{-1}\alpha^*\beta + q^{-1}(1-q^2)\gamma^*\delta,\\ \gamma\alpha^* = q\alpha^*\gamma, \quad \delta\alpha^* = \alpha^*\delta, \quad \gamma\beta^* = \beta^*\gamma,\\ \delta\beta^* = q\beta^*\delta - q(1-q^2)\alpha^*\gamma, \quad \delta\gamma^* = q^{-1}\gamma^*\delta,\\ \alpha\alpha^* = \alpha^*\alpha + (1-q^2)\gamma^*\gamma, \quad \beta\beta^* = \beta^*\beta + (1-q^2)(\delta^*\delta - \alpha^*\alpha) - (1-q^2)^2\gamma^*\gamma,\\ \gamma\gamma^* = \gamma^*\gamma, \quad \delta\delta^* = \delta^*\delta - (1-q^2)\gamma^*\gamma.\end{gathered} \tag{2.1}$$

The rest commutative relations can be read off from the rule $(ab)^* = b^*a^*$. We cast the generators into the matrix form

$$w = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad w^* = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}.$$

With the comultiplication $\Delta : \mathcal{A}_q \to \mathcal{A}_q \bigotimes \mathcal{A}_q$

$$\Delta\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \bigotimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

the antipode $S : \mathcal{A}_q \to \mathcal{A}_q$

$$S\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix},$$

and the counit $\epsilon : \mathcal{A}_q \to \mathbf{C}$

$$\epsilon\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathcal{A}_q$ becomes a Hopf algebra. In fact it is a *-Hopf algebra since

$$(\Delta(a))^* = \Delta(a^*)$$

and

$$S \circ * \circ S \circ * = id. \tag{2.2}$$

We define the *-Hopf subalgebra $\mathcal{A}_q(SU_2)$ by the generators

$$\mathcal{A}_q(SU_2) = \{\omega_c = \begin{pmatrix} \alpha_c & -q\gamma_c^* \\ \gamma_c & \alpha_c^* \end{pmatrix}\} \tag{2.3}$$

and the relations

$$\alpha_c^*\alpha_c + \gamma_c^*\gamma_c = 1, \quad \alpha_c\alpha_c^* + q^2\gamma_c^*\gamma_c = 1,$$
$$\gamma_c^*\gamma_c = \gamma_c\gamma_c^*, \quad \alpha_c\gamma_c^* = q\gamma_c^*\alpha_c, \quad \alpha_c\gamma_c = q\gamma_c\alpha_c.$$

Then

$$\omega_c^*\omega_c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.4}$$

In a similar way

$$\mathcal{A}_q(AN_q) = \{\omega_d = \begin{pmatrix} h & n \\ 0 & h^{-1} \end{pmatrix}\} \tag{2.5}$$

$$hh^* = h^*h, \quad hn = qnh, \quad hn^* = q^{-1}n^*h,$$
$$nn^* = n^*n + (1-q^2)((h^*h)^{-2} - 1).$$

The Iwasawa decomposition in the quantum context takes the form [2]

$$\omega = \omega_c\omega_d, \quad \omega \in \mathcal{A}_q(SL_2(\mathbf{C})), \omega_c \in \mathcal{A}_q(SU_2), \quad \omega_d \in \mathcal{A}_q(AN_q). \tag{2.6}$$

Natural description of commutative relations (2.1) can be obtained from the construction of the quantum double. It was implemented in [3],where $\mathcal{A}_q(SL_2(\mathbf{C}))$ is described as a special quantum double of $\mathcal{A}_q(SU_2)$, and (2.2) is derived by means of the corresponding $R$-matrix.

**Definition 2.1** *The quantum Lobachevsky space* $\mathbf{L}_q^3$ *is a *-subalgebra of* $\mathcal{A}_q(SL_2(\mathbf{C}))$ *generated by the bilinear constituents*

$$\omega^*\omega = \begin{pmatrix} \alpha^*\alpha + \gamma^*\gamma & \alpha^*\beta + \gamma^*\delta \\ \beta^*\alpha + \delta^*\gamma & \beta^*\beta + \delta^*\delta \end{pmatrix} = \begin{pmatrix} p & s \\ s^* & r \end{pmatrix} \tag{2.7}$$

Evidently, * acts as

$$p^* = p, \quad (s)^* = s^*, \quad r^* = r.$$

We don't need the explicit form of the commutative relations between $p, s, s^*$ and $r$ - they can be derived from (2.1).

Introduce a new generator $z$ instead of $n$

$$n = hz.$$

Then due to (2.4), (2.5) and (2.7)

$$p = H = h^*h = hh^*, \quad s = Hz, \quad s^* = z^*H, \quad r = z^*Hz + H^{-1}. \tag{2.8}$$

Consider now the complex associative algebra $U_q(SL_2(\mathbf{C}))$ with unit 1, generators $A, B, C, D$ and the relations

$$AD = DA = 1, \quad AB = qBA, \quad BD = qDB,$$
$$AC = q^{-1}CA, \quad CD = q^{-1}DC,$$
$$[B, C] = \frac{A^2 - D^2}{q - q^{-1}}. \tag{2.9}$$

In fact it is the Hopf algebra where

$$\Delta(A) = A \bigotimes A, \quad \Delta(D) = D \bigotimes D,$$

$$\Delta(B) = A \bigotimes B + B \bigotimes D, \Delta(C) = A \bigotimes C + C \bigotimes D, \tag{2.10}$$

$$\epsilon \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.11}$$

$$\epsilon \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D & -q^{-1}B \\ -qC & A \end{pmatrix}. \tag{2.12}$$

There exists a non-degenerate dilinear form $\langle u, a \rangle : U_q \times \mathcal{A}_q \to \mathbf{C}$ such that

$$\langle \Delta(u), a \bigotimes b \rangle = \langle u, ab \rangle, \quad \langle u \bigotimes v, \Delta(a) \rangle = \langle uv, a \rangle,$$

$$\langle 1_U, a \rangle = \epsilon_{\mathcal{A}}(a), \langle u, 1_{\mathcal{A}} \rangle = \epsilon_U(u), \langle S(u), a \rangle = \langle u, S(a) \rangle.$$

It takes the form of the generators

$$\langle A, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rangle = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix},$$

$$\langle D, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rangle = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix},$$

$$\langle B, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \langle C, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.13}$$

Moreover, $U_q(SL_2(\mathbf{C}))$ is the *-Hopf algebra in duality, where the involution is defined by the pairing

$$\langle u^*, a \rangle = \langle u, (\bar{S}(a))^* \rangle. \tag{2.14}$$

The element

$$\Omega_q = \frac{(q^{-1} + q)(A^2 + D^2) - 4}{2(q^{-1} - q)^2} + \frac{1}{2}(BC + CB) \tag{2.15}$$

is a Casimir element, since it commutes with any $u \in U_q(SL_2(\mathbf{C}))$.

The right action of $u \in U_q(SL_2(\mathbf{C}))$ on $\mathcal{A}$ is defined as [4]

$$a.u = (u \bigotimes id)(\Delta(a)). \tag{2.16}$$

It is the algebra action:

$$a.(uv) = (a.u).v \tag{2.17}$$

which satisfies the Leibnitz rule

$$(ab).u = \sum_j (a.u_j^1)(b.u_j^2) \tag{2.18}$$

where $\Delta(u) = \sum_j u_j^1 \otimes u_j^2$. The left action is defined in the same way.

The right action on the generators takes the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.A = \begin{pmatrix} q^{1/2}\alpha & q^{-1/2}\beta \\ q^{1/2}\gamma & q^{-1/2}\delta \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.B = \begin{pmatrix} 0 & \alpha \\ 0 & \gamma \end{pmatrix},$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.C = \begin{pmatrix} \beta & 0 \\ \delta & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.D = \begin{pmatrix} q^{-1/2}\alpha & q^{1/2}\beta \\ q^{-1/2}\gamma & q^{1/2}\delta \end{pmatrix}. \tag{2.19}$$

We will define now the right action of $U_q(SL_2(\mathbf{C}))$ on $\mathbf{L}_q^3$, which endows the latter with the structure of the right *-module. For any $a \in \mathbf{L}_q^3$ define the normal ordering using (2.1)

$$\ddagger a \ddagger = \sum_k c_k a_{1,k}^* a_{2,k} \tag{2.20}$$

where $a^*_{1,k}(a_{2,k})$ are monoms derpending ov $\alpha^*, \beta^*, \gamma^*, \delta^*(\alpha, \beta, \gamma, \delta)$. Then the right action on $\mathbf{L}^3_q$, which will de denoted as $(a).u$, is defined as follows

$$(a).u = \sum_k c_k a^*_{1,k}(a_{2,k}.u). \tag{2.21}$$

The generators $H, z$ are expressed by generators $\alpha, \ldots, \delta^*$ as

$$H = \alpha^*\alpha + \gamma^*\gamma, \tag{2.22}$$

$$z = \alpha^{-1}\beta + \sum_{k=0}^{\infty}(-1)^k q^{-2k}(y^*)^{k+1}y^k\alpha^{-2}, \quad y = \gamma\alpha^{-1}. \tag{2.23}$$

Let $w(m,r,n) = (z^*)^m H^r z^n$. Then using (2.19), (2.21) - (2.23) we can define the right action of generators $A, B, C, D$ on monoms $w(m,r,n)$.

$$w(m,r,n).A = q^{-n+\frac{r}{2}}w(m,r,n),$$

$$w(m,r,n).B = q^{-n+\frac{r+1}{2}}\frac{1-q^{2n}}{1-q^2}w(m,r,n-1),$$

$$w(m,r,n).C = q^{n+\frac{r-1}{2}}\frac{1-q^{2m}}{1-q^2}w(m-1,r-2,n) - q^{-n+\frac{r+3}{2}}\frac{1-q^{2n-2r}}{1-q^2}w(m,r,n+1),$$

$$w(m,r,n).D = q^{n-\frac{r}{2}}w(m,r,n). \tag{2.24}$$

The second Casimir (2.15) acts on monom $w(m,r,n)$ as

$$w(m,r,n).\Omega_q = q^{1-r}\left(\frac{1-q^{r+1}}{1-q^2}\right)^2 w(m,r,n) + q^{r-1}\frac{(1-q^{2m})(1-q^{2n})}{(1-q^2)^2}w(m-1,r-2,n-1). \tag{2.25}$$

**Remark 2.1**

$$\lim_{q\to 1-0}\partial_q A = d_A, \qquad \lim_{q\to 1-0}\partial_q D = d_D,$$

$$\lim_{q\to 1-0} B = d_B, \qquad \lim_{q\to 1-0} C = d_C,$$

*and*

$$\lim_{q\to 1-0}\Omega_q = \frac{1}{2}\Omega + \frac{1}{4}.$$

## 3 Modified $q$-Bessel Functions

We remind the fundamental formulas from the theory of the basic hypergeometrical serieses. For any $q \in (0,1)$

$$(a,q)_n = \begin{cases} 1 & for \quad n=0 \\ (1-a)(1-aq)\ldots(1-aq^{n-1}) & for \quad n\geq 1, \end{cases}$$

$$(a,q)_\infty = \lim_{n\to\infty}(a,q)_n, \quad (a_1,\ldots,a_k,q)_\infty = (a_1,q)_\infty\ldots(a_k,q)_\infty. \tag{3.1}$$

The $q-\Gamma$-function is defined in following way

$$\Gamma_q(\nu) = \frac{(q,q)_\infty}{(q^\nu,q)_\infty}(1-q)^{1-\nu}.$$

The basic hypergeometrical funcyion is

$${}_r\Phi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;q,x) = \sum_{n=0}^{\infty}\frac{(a_1,q)_n\ldots(a_r,q)_n}{(q,q)_n(b_1,q)_n\ldots(b_s,q)_n}[(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r}x^n \tag{3.2}$$

The $q$-exponents are

$$e_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{(q,q)_n}, \qquad |x|<1, \tag{3.3}$$

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{(q,q)_n}. \tag{3.4}$$

The $q$-exponents can be represented by forms:

$$e_q(x) = \frac{1}{(x,q)_\infty}, \qquad E_q(x) = (-x,q)_\infty. \tag{3.5}$$

There are two typies of $q$-trigonometric functions

$$\cos_q x = \frac{1}{2}[e_q(ix) + e_q(-ix)], \quad \sin_q x = \frac{1}{2i}[e_q(ix) - e_q(-ix)], \tag{3.6}$$

$$\mathrm{Cos}_q\, x = \frac{1}{2}[E_q(ix) + E_q(-ix)], \quad \mathrm{Sin}_q\, x = \frac{1}{2i}[E_q(ix) - E_q(-ix)]. \tag{3.7}$$

Let consider the complete elliptic integrals

$$\mathbf{K}(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2\alpha}}, \qquad \mathbf{K}'(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{\cos^2\alpha + k^2\sin^2\alpha}},$$

and let

$$\ln q = -\frac{\pi \mathbf{K}'(k)}{\mathbf{K}(k)}. \tag{3.8}$$

Then [6](5.3.6.1)

$$\mathbf{Q}_\nu = (1-q)\sum_{m=-\infty}^{\infty} \frac{1}{q^{m-\nu+\frac{1}{2}} + q^{-m+\nu-\frac{1}{2}}} = \frac{1-q}{\pi}\mathbf{K}(k)\,\mathrm{dn}[\frac{2\ln q^{-\nu+\frac{1}{2}}}{\pi}\mathbf{K}'(k)]. \tag{3.9}$$

where $\mathrm{dn}\, u = \sqrt{1-k^2\sin^2\phi}$, $u = \int_0^\phi \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$ ($\mathrm{dn}\, u$ is the Jacobi elliptic function). If $u = 0$ then $\phi = 0$ and $\mathrm{dn}\, u = 1$. It followes from (3.8) and (3.9) for any $\nu$

$$\lim_{q\to 1-0} \mathbf{Q}_\nu = \frac{\pi}{2}.$$

In [1] the $q$-Bessel function were determined as

$$J_\nu^{(1)}(s;q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}(s/2)^\nu\ {}_2\Phi_1(0,0;q^{\nu+1};q,-\frac{s^2}{4}),$$

$$J_\nu^{(2)}(s;q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}(s/2)^\nu\ {}_0\Phi_1(-;q^{\nu+1};q,-\frac{s^2 q^{\nu+1}}{4})$$

where ${}_r\Phi_s$ is determined by formula (2.1). We will define the modified $q$-Bessel function $I_\nu^{(j)}(s;q)$ so that the following condition is fulfilled

$$I_\nu^{(j)}(s;q) = e^{-i\frac{\nu\pi}{2}} J_\nu^{(j)}(e^{i\frac{\pi}{2}}s;q), \qquad j = 1,2 \tag{3.10}$$

We will consider below the functions

$$I_\nu^{(1)}((1-q^2)s;q^2) = \frac{1}{\Gamma_{q^2}(\nu+1)}\sum_{n=0}^{\infty}\frac{(1-q^2)^{2n}s^{\nu+2n}}{(q^2,q^2)_n(q^{2\nu+2},q^2)_n 2^{\nu+2n}}, \quad |s| < \frac{1}{1-q^2}, \tag{3.11}$$

$$I_\nu^{(2)}((1-q^2)s;q^2) = \frac{1}{\Gamma_{q^2}(\nu+1)}\sum_{n=0}^{\infty}\frac{q^{2n(\nu+n)}(1-q^2)^{2n}s^{\nu+2n}}{(q^2,q^2)_n(q^{2\nu+2},q^2)_n 2^{\nu+2n}}. \tag{3.12}$$

If $|q| < 1$, the series (3.12) converges for all $s \neq 0$. Therefore $I_\nu^{(2)}((1-q^2)s;q^2)$ is holomorphic function outside of $z = 0$.

**Remark 3.1**

$$\lim_{q\to 1-0} I_\nu^{(j)}((1-q^2)s;q^2) = I_\nu(s), \quad j = 1,2$$

The function $I_\nu^{(1)}((1-q^2)s;q^2)$ is the meromorphic function outside of $z=0$ with the ordinary poles in the points $s \neq \pm\frac{2q^{-r}}{1-q^2}, r=0,1,\ldots$

**Remark 3.2** *If $q \to 1-0$ the all poles of $I_\nu^{(1)}((1-q^2)s;q^2)$*

$$s \neq \pm\frac{2q^{-r}}{1-q^2}, \qquad r=0,1,\ldots$$

*go to infinity along the real axis.*

We have from (3.11) immediately

**Proposition 3.1** *The function $I_\nu^{(1)}((1-q^2)s;q^2)$ satisfies the following relations*

$$\frac{2\partial_s}{1+q}s^\nu I_{-\nu}^{(1)}((1-q^2)s;q^2) = s^{\nu-1}I_{-\nu+1}^{(1)}((1-q^2)s;q^2),$$

$$\frac{2\partial_s}{1+q}s^\nu I_{\nu}^{(1)}((1-q^2)s;q^2) = s^{\nu-1}I_{\nu-1}^{(1)}((1-q^2)s;q^2),$$

*and the difference equation*

$$[1-(\frac{1-q^2}{2})^2q^{-2}s^2]f(q^{-1}s)-(q^\nu+q^{-\nu})f(s)+f(qs)=0 \tag{3.13}$$

Analogy we have from (3.12)

**Proposition 3.2** *The function $I_\nu^{(2)}((1-q^2)s;q^2)$ satisfies the relations*

$$\frac{2\partial_s}{1+q}s^\nu I_{-\nu}^{(2)}((1-q^2)s;q^2) = q^{-\nu+1}s^{\nu-1}I_{-\nu+1}^{(2)}((1-q^2)qs;q^2),$$

$$\frac{2\partial_s}{1+q}s^\nu I_{\nu}^{(2)}((1-q^2)s;q^2) = q^{-\nu+1}s^{\nu-1}I_{\nu-1}^{(2)}((1-q^2)qs;q^2),$$

*and the difference equation*

$$f(q^{-1}s)-(q^\nu+q^{-\nu})f(s)+[1-(\frac{1-q^2}{2})^2s^2]f(qs)=0 \tag{3.14}$$

It is easy to show that the functions (3.11) and (3.12) are connected by correlations:

$$I_\nu^{(1)}((1-q^2)s;q^2) = e_{q^2}(\frac{(1-q^2)^2}{4}z^2)I_\nu^{(2)}((1-q^2)s;q^2) \tag{3.15}$$

$$I_\nu^{(2)}((1-q^2)s;q^2) = E_{q^2}(-\frac{1-q^2}{2}s^2)I_\nu^{(1)}((1-q^2)s;q^2) \tag{3.16}$$

# 4 The $q$-Bessel-Macdonald Function

Unfortunately the function $I_\nu^{(1)}((1-q^2)s;q^2)$ is determined by power series (3.11) in domain $|s|<\frac{2}{1-q^2}$ only while we need a representation of this function as series on the whole complex plane. But the following proprsition takes place

**Proposition 4.1** *The function $I_\nu^{(1)}((1-q^2)s;q^2)$ for $s\neq 0$ can be represented as*

$$I_\nu^{(1)}((1-q^2)s;q^2) = \frac{a_\nu}{\sqrt{s}}[e_q(\frac{1-q^2}{2}z)\Phi_\nu(s)+ie^{i\nu\pi}e_q(-\frac{1-q^2}{2}z)\Phi_\nu(-s)], \tag{4.1}$$

*where*

$$\Phi_\nu(s) = {}_2\Phi_1(q^{\nu+1/2},q^{-\nu+1/2};-q;q,\frac{2q}{(1-q^2)s}), \tag{4.2}$$

*and*

$$a_\nu = \sqrt{\frac{2}{1-q^2}}e_q(-1)\frac{I_\nu^{(2)}(2;q^2)}{\Phi_\nu(\frac{2}{1-q^2})} \tag{4.3}$$

The coefficients $a_\nu$ (4.3) satisfy the recurrent relation

$$a_{\nu+1} = a_\nu q^{-\nu-1/2}$$

and the condition

$$a_\nu a_{-\nu} = \frac{q^{-\nu+1/2}}{2\Gamma_{q^2}(\nu)\Gamma_{q^2}(1-\nu)\sin\nu\pi}.$$

**Proposition 4.2** *The function $I_\nu^{(2)}((1-q^2)s;q^2)$ for $s \neq 0$ can be represented by*

$$I_\nu^{(2)}((1-q^2)s;q^2) = \frac{a_\nu}{\sqrt{s}}[e_{q^2}(\frac{(1-q^2)^2}{4}z^2)\Phi_\nu(s) + ie^{i\nu\pi}E_q(-\frac{1-q^2}{2}s)\Phi_\nu(-s)], \quad (4.4)$$

In the classical analyses the Bessel-Macdonald function is defined as

$$K_\nu(s) = \frac{\pi}{2\sin\nu\pi}[I_{-\nu}(s) - I_\nu(s)] \quad (4.5)$$

for $\nu \neq n$, and if $\nu = n$ by the limit for $\nu \to n$ in (4.5). Here we present the correct "quantization" of thie definition in such way that other properties are also quantized in consistent way.

**Definition 4.1** *The q-Bessel-Maclonald functions (q-BMF) are defined as*

$$K_\nu^{(j)}((1-q^2)s;q^2) = \frac{1}{2}q^{-\nu^2+\nu}\Gamma_{q^2}(\nu)\Gamma_{q^2}(1-\nu)[\sqrt{\frac{a_\nu}{a_{-\nu}}}I_{-\nu}^{(j)}((1-q^2)s;q^2) - \sqrt{\frac{a_{-\nu}}{a_\nu}}I_\nu^{(j)}((1-q^2)s;q^2)], \quad (4.6)$$

*with $a_\nu$ (4.3), $j = 1,2$.*

As in the classical case this definition should be adjusted for the integer values of the index $\nu = n$ by the limit for $\nu \to n$ in (4.6)

It follows from (4.1), (4.4) and (4.6)

$$K_\nu^{(1)}((1-q^2)s;q^2) = \frac{q^{-\nu^2+1/2}}{2\sqrt{a_\nu a_{-\nu}}\sqrt{s}}e_q(-\frac{1-q^2}{2}s)\Phi_\nu(-s), \quad (4.7)$$

$$K_\nu^{(2)}((1-q^2)s;q^2) = \frac{q^{-\nu^2+1/2}}{2\sqrt{a_\nu a_{-\nu}}\sqrt{s}}E_q(-\frac{1-q^2}{2}s)\Phi_\nu(-s). \quad (4.8)$$

It is easu to prove using (4.7) and (4.8) the following

**Proposition 4.3** *q-BMF $K_\nu^{(1)}((1-q^2)s;q^2)$ is a holomorphic function in domain $\mathrm{Re}\, s > \frac{2q}{1-q^2}$.*

**Proposition 4.4** *q-BMF $K_\nu^{(2)}((1-q^2)s;q^2)$ is a holomorphic function in the domain $s \neq 0$.*

The next propositions take place

**Proposition 4.5** *The function $K_\nu^{(1)}((1-q^2)s;q^2)$ satisfies the following relations*

$$\frac{2\partial_s}{1+q}s^\nu K_\nu^{(1)}((1-q^2)s;q^2) = -s^{\nu-1}K_{\nu-1}^{(1)}((1-q^2)s;q^2),$$

$$\frac{2\partial_s}{1+q}s^{-\nu} K_\nu^{(1)}((1-q^2)s;q^2) = -s^{-\nu-1}K_{\nu+1}^{(1)}((1-q^2)s;q^2)$$

*and the difference equation (3.13).*

**Proposition 4.6** *The function $K_\nu^{(2)}((1-q^2)s;q^2)$ satisfies the following relations*

$$\frac{2\partial_s}{1+q}s^\nu K_\nu^{(2)}((1-q^2)s;q^2) = -q^{-\nu+1}s^{\nu-1}K_{\nu-1}^{(2)}((1-q^2)s;q^2),$$

$$\frac{2\partial_s}{1+q}s^{-\nu} K_\nu^{(2)}((1-q^2)s;q^2) = -q^{-\nu+1}s^{-\nu-1}K_{\nu+1}^{(2)}((1-q^2)s;q^2)$$

*and the difference equation (3.14).*

**Proposition 4.7** *For any $\nu$ the functions $I_\nu^{(1)}((1-q^2)s;q^2)$ and $K_\nu^{(1)}((1-q^2)s;q^2)$ form a fundamential system of solutions to the equation (3.13).*

**Proposition 4.8** *For any $\nu$ the functions $I_\nu^{(2)}((1-q^2)s;q^2)$ and $K_\nu^{(2)}((1-q^2)s;q^2)$ form a fundamential system of solutions to the equation (3.14).*

**Remark 4.1**

$$\lim_{q\to 1-0} K_\nu^{(j)}((1-q^2)s;q^2) = K_\nu(s), \qquad j=1,2.$$

**Remark 4.2** *If $q \to 1-0$ the representations (4.1), (4.4), (4.7) and (4.8) give us the well-known asymptotic decompositions for the functions $I_\nu(s)$ and $K_\nu(s)$ respectivety [7].*

# 5 The Jackson Integral Representation of the Modified $q$-Bessel Functions and $q$-Bessel-Macdonald Functions

Jackson $q$-integral is determined as the map an algebra of functions of one variable into a set of the number serieses

$$\int_{-1}^{1} f(x)d_qx = (1-q)\sum_{m=0}^{\infty} q^m[f(q^m)+f(-q^m)],$$

$$\int_{0}^{\infty} f(x)d_qx = (1-q)\sum_{m=-\infty}^{\infty} q^m f(q^m),$$

$$\int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{-\infty}^{\infty} q^m[f(q^m)+f(-q^m)].$$

Define the difference operator

$$\partial_x f(x) = \frac{x^{-1}}{1-q}[f(x)-f(qx)]. \tag{5.1}$$

The following formulas of the $q$-integration by parts are valid

$$\int_{-1}^{1}\phi(x)\partial_x\psi(x)d_qx = \phi(1)\psi(1)-\phi(-1)\psi(-1)-\int_{-1}^{1}\partial_x\phi(x)\psi(qx)d_qx, \tag{5.2}$$

$$\int_{0}^{\infty}\phi(x)\partial_x\psi(x)d_qx = \lim_{m\to\infty}[\phi(q^{-m})\psi(q^{-m})-\phi(q^m)\psi(q^m)]-\int_{0}^{\infty}\partial_x\phi(x)\psi(qx)d_qx. \tag{5.3}$$

$$\int_{-\infty}^{\infty}\phi(x)\partial_x\psi(x)d_qx = \lim_{m\to\infty}[\phi(q^{-m})\psi(q^{-m})+\phi(-q^{-m})\psi(-q^{-m})]-\int_{-\infty}^{\infty}\partial_x\phi(x)\psi(qx)d_qx. \tag{5.4}$$

The last two expressions imply the regularizations of the improper integrals.

Let $z$ and $s$ be noncommuting elements and

$$zs = qsz. \tag{5.5}$$

Consider the function

$$f(x) = \sum_r a_r x^r. \tag{5.6}$$

*The rule of $q$-integration* in the noncommutative case

$$\int f(zs)d_qs = \int\sum_r a_r(zs)^r d_qs = \int\sum_r a_r q^{-\frac{r(r-1)}{2}} z^r s^r d_qs,$$

$$\int d_qz f(zs) = \int d_qz\sum_r a_r(zs)^r = \int d_qz\sum_r a_r q^{-\frac{r(r-1)}{2}} z^r s^r.$$

Define the following transformation $\ddagger f\ddagger$ for functions $f$ depending on the noncommutative variables $s$ and $z$ (5.5). If we have function which has the form (5.6) and all monoms are order we will write

$$f(zs) = \sum_r a_r(zs)^r \to \ddagger f(zs)\ddagger = \sum_r a_r z^r s^r.$$

**Definition 5.1** *The function $f(z)$ is absolutely $q$-integrable if the series*

$$\sum_{m=-\infty}^{\infty} q^m f(q^m)$$

*converges absolutely.*

It means, in particular, that

$$\lim_{m\to\pm\infty} q^m |f(q^m)| = 0$$

It follows from (3.2) - (3.4)

**Proposition 5.1**

$$_0\Phi_1(-;0;q,\frac{1-q^2}{2}zs) = \ddagger E_q(\frac{1-q^2}{2}zs)\ddagger, \tag{5.7}$$

$$E_q(\frac{1-q^2}{2}zs) = \ddagger e_q(\frac{1-q^2}{2}zs)\ddagger. \tag{5.8}$$

There is a $q$-analog of classical binomial formula [5]

$$(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}, \qquad |z| < 1,$$

$$\frac{(q^\alpha z, q)_\infty}{(z,q)_\infty} = \sum_{k=0}^{\infty} \frac{(q^\alpha, q)_k}{(q,q)_k} z^k, \qquad |z| < 1.$$

We need in two generalizations of this $q$-binom

$$r(a,b,z,q) = \frac{(az,q)_\infty}{(bz,q)_\infty} \tag{5.9}$$

$$R(a,b,\gamma,z,q^2) = \frac{(az^2,q^2)_\infty}{(bz^2,q^2)_\infty} z^\gamma \tag{5.10}$$

The function (5.9) satisfies the difference equation

$$z[br(a,b,z,q) - ar(a,b,qz,q)] = r(a,b,z,q) - r(a,b,qz,q). \tag{5.11}$$

The function (5.10) satisfies the difference equation

$$z^2[bq^\gamma R(a,b,\gamma,z,q^2) - aR(a,b,\gamma,qz,q^2)] = q^\gamma R(a,b,\gamma,z,q^2) - R(a,b,\gamma,qz,q^2). \tag{5.12}$$

**Lemma 5.1** *If $|a| < |b|$ the function $r(a,b,z,q)$ can be represented as the sum of the partial functions*

$$\frac{(az,q)_\infty}{(bz,q)_\infty} = \frac{1}{(q,q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k+1)}{2}} (a/bq^{-k}, q)_\infty}{(q,q)_k (1 - zbq^k)}. \tag{5.13}$$

*The series (5.13) converges absolutely for any $z \neq b^{-1}q^{-k}$, $\quad k = 0, 1, \ldots$*

**Remark 5.1** *If $0 < |a| < |b|$ then*

$$\frac{(az,q)_\infty}{(bz,q)_\infty} = \frac{(a/b,q)_\infty}{(q,q)_\infty} \sum_{k=0}^{\infty} \frac{(b/aq,q)_k (a/b)^k}{(q,q)_k (1 - zbq^k)}. \tag{5.14}$$

*If $a = 0$ then*

$$\frac{1}{(bz,q)_\infty} = \frac{1}{(q,q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q,q)_k (1 - zbq^k)}. \tag{5.15}$$

Assume that $a = \epsilon q^{2\alpha}, b = \epsilon q^{2\beta}, \epsilon = \pm 1$ in (5.10). Then we have from (5.14)

**Corollary 5.1**

$$z^{\gamma}\frac{(\epsilon q^{2\alpha}z^2,q^2)_\infty}{(\epsilon q^{2\beta}z^2,q^2)_\infty}=z^{\gamma}\frac{(q^{2(\alpha-\beta)},q^2)_\infty}{(q^2,q^2)_\infty}\sum_{k=0}^{\infty}\frac{(q^{2(\beta-\alpha+1)},q^2)_k q^{2(\alpha-\beta)k}}{(q^2,q^2)_k(1-\epsilon z^2q^{2(\beta+k)})}. \qquad (5.16)$$

**Remark 5.2** *As it follows from [5](1.3.2)*

$$\frac{(\epsilon q^{2\alpha}z^2,q^2)_\infty}{(\epsilon q^{2\beta}z^2,q^2)_\infty}=\sum_{k=0}^{\infty}\epsilon^k q^{2\beta k}\frac{(q^{2(\alpha-\beta)},q^2)_k}{(q^2,q^2)_k}z^{2k}. \qquad (5.17)$$

*which converges in the domain $|z|<q^{-\beta/2}$.*

It follows from Lemma 5.1 that if $\gamma=0$ (5.16) is the meromorphic function with the ordinary poles $z=\pm\sqrt{\epsilon}q^{-\beta-k}$, $k=0,1,\ldots$, and hence it is the analitic continuation of (5.17).

**Corollary 5.2** *For an arbitrary real $s\neq 0$*

$$\lim_{m\to\infty}|e_q(i\frac{1-q^2}{2}q^{-m}s)|=0.$$

**Corollary 5.3** *For real $s\neq 0$ and integer $m$*

$$|\cos_q(\frac{1-q^2}{2}q^{-m}s)|\leq\frac{E_q(q^{-1})e_q(q)}{1+(\frac{1-q^2}{2})^2q^{-2m}s^2}.$$

**Corollary 5.4** *If $\alpha>\beta+1$ and real $z\neq 0$, then*

$$\frac{(-q^{2\alpha}z^2,q^2)_\infty}{(-q^{2\beta}z^2,q^2)_\infty}\leq\frac{C_{\alpha,\beta}}{1+z^2q^{2\beta}}.$$

**Remark 5.3** *Let $a=\epsilon q^{2\alpha}, b=\epsilon q^{2\beta}$ in (5.10) and (5.12). Then if $q\to 1-0$ the difference equation (5.12) takes the form of the differential equation*

$$z(1-\epsilon z^2)R'(z)-[\gamma+\epsilon(2\alpha-2\beta-\gamma)z^2]R(z)=0 \qquad (5.18)$$

*with solution*

$$R(z)=Cz^{\gamma}(1-\epsilon z^2)^{\beta-\alpha}.$$

**Proposition 5.2** *Modified q-Bessel function (q-MBF) $I_\nu^{(1)}$ for $\nu>0$ can be represented as the q-integral*

$$I_\nu^{(1)}((1-q^2)s;q^2)=\frac{1+q}{2\Gamma_{q^2}(\nu+1/2)\Gamma_{q^2}(1/2)}\int_{-1}^{1}d_qz\frac{(q^2z^2,q^2)_\infty}{(q^{2\nu+1}z^2,q^2)_\infty}E_q(\frac{1-q^2}{2}zs)(s/2)^{\nu}. \qquad (5.19)$$

**Proof.** Consider the $q$-integral

$$S_1^{(1)}(s)=\int_{-1}^{1}d_qzf_\nu^{(1)}(z)E_q(\frac{1-q^2}{2}zs), \qquad (5.20)$$

where $g_\nu^{(1)}(z)$ is such function that it is absolutely convergent. Require that $S_1^{(1)}(s)(s/2)^\nu$ satisfies (3.13). Then $S_1^{(1)}(s)$ satisfies the equation

$$S_1^{(1)}(q^{-1}s)-S_1^{(1)}(s)-q^{2\nu}[S_1^{(1)}(s)-S_1^{(1)}(qs)]=(\frac{1-q^2}{2})^2q^{-2}S_1^{(1)}(q^{-1}s)s^2. \qquad (5.21)$$

Substituting (5.20) in (5.21), using the rule of $q$-integration, (5.8) and (5.2), we come to the difference equation for $f_\nu^{(1)}(z)$

$$q^{2\nu+1}z^2[f_\nu^{(1)}(z)-q^{-2\nu+1}f_\nu^{(1)}(qz)]=f_\nu^{(1)}(z)-f_\nu^{(1)}(qz). \qquad (5.22)$$

It coincides with (5.12) for $a=q^2, b=q^{2\nu+1},\gamma=0$, and hence

$$f_\nu^{(1)}(z)=\frac{(q^2z^2,q^2)_\infty}{(q^{2\nu+1}z^2,q^2)_\infty}. \qquad (5.23)$$

$S_1^{(1)}(s)(s/2)^\nu$ is a solution to (3.13) and therefore it can be represented as (see Proposition 4.7)

$$S_1^{(1)}(s)(s/2)^\nu = AI_\nu^{(1)}((1-q^2)s;q^2) + BK_\nu^{(1)}((1-q^2)s;q^2).$$

Multiplying the both sides on $(s/2)^\nu$ and putting $s=0$ from (3.11) ( (4.6) we obtain $B=0$. Multiplying again on $(s/2)^{-\nu}$ and assuming $s=0$ we come to

$$\int_{-1}^{1} d_q z f_\nu^{(1)}(z) = A\frac{1}{\Gamma_{q^2}(\nu+1)}.$$

It follows from [5] (1.11.7)

$$A = \frac{2}{1+q}B_{q^2}(\nu+1/2,1/2)\Gamma_{q^2}(\nu+1) = \frac{2}{1+q}\Gamma_{q^2}(\nu+1/2)\Gamma_{q^2}(1/2).$$

and we come to (5.19).■ At the same way we can prove the following

**Proposition 5.3** *The q-MBF $I_\nu^{(2)}((1-q^2)s;q^2)$ for $\nu>0$ has the following q-integral representation*

$$I_\nu^{(2)}((1-q^2)s;q^2) = \frac{1+q}{2\Gamma_{q^2}(\nu+1/2)\Gamma_{q^2}(1/2)}\times$$

$$\times\int_{-1}^{1} d_q z\frac{(q^2z^2,q^2)_\infty}{(q^{2\nu+1}z^2,q^2)_\infty}\,{}_0\Phi_1(-;0;q,\frac{1-q^2}{2}q^{\nu+1/2}zs)(s/2)^\nu. \tag{5.24}$$

**Remark 5.4** *If $q\to 1-0$ the equation (5.22) takes the form of the differential equation (see Remark 5.3)*

$$(1-z^2)f_\nu'(z) + (2\nu-1)zf_\nu(z) = 0.$$

*The solution to this equation is*

$$f_\nu(z) = C(1-z^2)^{\nu-1/2},$$

*which leads to the classical integral representation of Modified Bessel function [7] (7.12.10)*

$$I_\nu(s) = \frac{(s/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)}\int_{-1}^{1}(1-z^2)^{\nu-1/2}e^{zs}dz.$$

**Proposition 5.4** *The q-BMF $K_\nu^{(1)}((1-q^2)s;q^2)$ for $\nu>0$ can be represented by the q-integral*

$$K_\nu^{(1)}((1-q^2)s;q^2) = \frac{q^{-\nu^2+1/2}\Gamma_{q^2}(\nu+1/2)\Gamma_{q^2}(1/2)}{4\mathbf{Q}_\nu}\sqrt{\frac{a_\nu}{a_{-\nu}}}\times$$

$$\times\int_{-\infty}^{\infty} d_q z\frac{(-q^2z^2,q^2)_\infty}{(-q^{-2\nu+1}z^2,q^2)_\infty}E_q(i\frac{1-q^2}{2}zs)(s/2)^{-\nu}, \tag{5.25}$$

*where $\mathbf{Q}_\nu$ is defined by (3.9).*

**Proposition 5.5** *The q-BMF $K_\nu^{(2)}((1-q^2)s;q^2)$ for $\nu>3/2$ can be represented by the q-integral*

$$K_\nu^{(2)}((1-q^2)s;q^2) = \frac{q^{-\nu^2+\nu}\Gamma_{q^2}(\nu+1/2)\Gamma_{q^2}(1/2)}{4\mathbf{Q}_{1/2}}\sqrt{\frac{a_\nu}{a_{-\nu}}}\times$$

$$\times\int_{-\infty}^{\infty} d_q z\frac{(-q^{2\nu+1}z^2,q^2)_\infty}{(-z^2,q^2)_\infty}\,{}_0\Phi_1(-;0;q,i\frac{1-q^2}{2}zs)(s/2)^{-\nu}, \tag{5.26}$$

*where $\mathbf{Q}_{1/2}$ is defined by (3.9).*

**Remark 5.5** *If $q\to 1-0$ the representations (5.25) and (5.26) give us the classical integral representation of Bessel-Macdonald function (the Fourier integral) [7] (7.12.27)*

$$K_\nu(s) = \frac{\Gamma(\nu+1/2)(s/2)^{-\nu}}{2\Gamma(1/2)}\int_{-\infty}^{\infty}(z^2+1)^{-\nu-1/2}e^{izs}dz.$$

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